

Sublinear separators, fragility and subexponential expansion

Zdeněk Dvořák*

Abstract

Let \mathcal{G} be a subgraph-closed graph class with bounded maximum degree. We show that if \mathcal{G} has balanced separators whose size is smaller than linear by a polynomial factor, then \mathcal{G} has subexponential expansion. This gives a partial converse to a result of Nešetřil and Ossona de Mendez. As an intermediate step, the proof uses a new kind of graph decompositions.

The concept of graph classes with bounded expansion was introduced by Nešetřil and Ossona de Mendez [31] as a way of formalizing the notion of sparse graph classes. Let us give a few definitions.

For a graph G , a k -minor of G is any graph obtained from G by contracting pairwise vertex-disjoint subgraphs of radius at most k and removing vertices and edges. Thus, a 0-minor is just a subgraph of G . Let us define $\nabla_k(G)$ as

$$\max \left\{ \frac{|E(G')|}{|V(G')|} : G' \text{ is a } k\text{-minor of } G \right\}.$$

For a class \mathcal{G} , let $\nabla_k(\mathcal{G})$ be the supremum of $\nabla_k(G)$ for $G \in \mathcal{G}$ (or ∞ if ∇_k is unbounded for the graphs in the class). If $\nabla_k(\mathcal{G})$ is finite for every $k \geq 0$, we say that \mathcal{G} has *bounded expansion*; and if f is a function such that $f(k) \geq \nabla_k(\mathcal{G})$ for every $k \geq 0$, we say that f *bounds the expansion of \mathcal{G}* . If $\lim_{k \rightarrow \infty} \frac{\log \nabla_k(\mathcal{G})}{k} = 0$, we say that \mathcal{G} has *subexponential expansion*.

The definition is quite general—examples of classes of graphs with bounded expansion include proper minor-closed classes of graphs, classes of graphs

*Computer Science Institute, Charles University, Prague, Czech Republic. E-mail: rakdver@iuuk.mff.cuni.cz. Supported by project GA14-19503S (Graph coloring and structure) of Czech Science Foundation and by project LH12095 (New combinatorial algorithms - decompositions, parameterization, efficient solutions) of Czech Ministry of Education.

with bounded maximum degree, classes of graphs excluding a subdivision of a fixed graph, classes of graphs that can be embedded on a fixed surface with bounded number of crossings per edge and many others, see [35].

Importantly, bounded expansion also implies a wide range of interesting structural and algorithmic properties, generalizing many results from proper minor-closed classes of graphs. For example, graphs in any class with bounded expansion have bounded chromatic number, acyclic chromatic number, star chromatic number, and other generalized variants of the chromatic number [31]. For graphs from such a class, there exists a linear-time algorithm to test the presence of a fixed subgraph [32] (as the subgraph testing problem is $W[1]$ -complete when parameterized by the subgraph [9], such an algorithm is unlikely to exist for all graphs). This algorithm was further generalized to testing any property expressible in the first order logic [13]. Other related results include bounds on the growth function of classes with bounded expansion [15], and parameterized algorithmic results on induced matchings [34] and dominating sets [10]. For a more in-depth introduction to the topic, the reader is referred to the book of Nešetřil and Ossona de Mendez [33].

The bounds and the time complexity of the algorithms we mentioned in the previous paragraph of course depend on the function bounding the expansion of the class; hence, it would be useful to be able to estimate this function for a given graph class. However, while there is an extensive theory for qualitatively deciding whether a class of graphs has bounded expansion [12, 35], we only know a tight estimate for the function bounding the expansion for a few special classes of graphs (proper minor-closed classes, and the class of graphs with given maximum degree).

One of the properties of graph classes with bounded expansion that might lead to improving the estimates is a connection to small balanced separators. A *separation* of a graph G is a pair (A, B) of edge-disjoint subgraphs of G such that $A \cup B = G$, and the *size* of the separation is $|V(A) \cap V(B)|$. Observe that G has no edge with one end with $V(A) \setminus V(B)$ and the other end in $V(B) \setminus V(A)$, and thus the set $V(A) \cap V(B)$ separates $V(A) \setminus V(B)$ from $V(B) \setminus V(A)$ in G . A separation (A, B) is *balanced* if $|V(A) \setminus V(B)| \leq 2|V(G)|/3$ and $|V(B) \setminus V(A)| \leq 2|V(G)|/3$. Note that $(G, G - E(G))$ is a balanced separation. For a graph class \mathcal{C} , let $s_{\mathcal{C}}(n)$ denote the smallest nonnegative integer such that every graph in \mathcal{C} with at most n vertices has a balanced separation of size at most $s_{\mathcal{C}}(n)$. We say that \mathcal{C} has *sublinear separators* if $\lim_{n \rightarrow \infty} \frac{s_{\mathcal{C}}(n)}{n} = 0$, and that \mathcal{C} has *strongly sublinear separators* if there exist constants $c \geq 1$ and $0 \leq \delta < 1$ such that

$s_{\mathcal{C}}(n) \leq cn^\delta$ for every $n \geq 0$.

Lipton and Tarjan brought focus on the notion of sublinear separators by showing in [28] that the class \mathcal{C}_p of planar graphs satisfies $s_{\mathcal{C}_p}(n) = O(\sqrt{n})$, and by pointing out that sublinear separators lead to a natural divide-and-conquer approach, useful especially in the design of efficient polynomial-time algorithms, as well as of approximation algorithms and of exact algorithms with subexponential time complexity [29]. Since then, numerous similar applications were found [19, 22, 17, 18, 6, 41, 25], establishing the importance of the concept.

Later, it was shown that graphs embedded on other surfaces [21] and all proper minor-closed graph classes [24] also have strongly sublinear separators. Building upon the previous result of Plotkin, Rao and Smith [37], Nešetřil and Ossona de Mendez [32] made the following observation linking balanced separators to bounded expansion, which qualitatively generalizes all the previous results (let us remark that the classes studied in [28, 21, 24] all have expansion bounded by a constant function).

Theorem 1 (Nešetřil and Ossona de Mendez [32, Theorem 8.3]). *Every graph class with subexponential expansion has sublinear separators.*

Theorem 1 can be used to establish a lower bound on the expansion function of a class, and it cannot be significantly improved, since 3-regular expanders have expansion bounded by $f(k) = 2^k$ and do not have sublinear separators. In this paper, we indicate that Theorem 1 might actually be an almost precise characterization of classes of graphs with sublinear separators (or, alternatively, of classes of graphs with subexponential expansion) by proving its weak converse.

Theorem 2. *Let \mathcal{G} be a subgraph-closed class of graphs with bounded maximum degree. If \mathcal{G} has strongly sublinear separators, then there exists $\gamma \geq 0$ such that the expansion of \mathcal{G} is bounded by $f(k) = \gamma e^{k^{3/4}}$. Hence, \mathcal{G} has subexponential expansion.*

Theorem 2 is the first general criterion implying subexponential expansion that has been found so far, and indeed, one of the first results giving a reasonably small upper bound on the expansion function of a class of graphs. The assumption that \mathcal{G} is subgraph-closed is natural, excluding dense graphs with balanced separators (such as two cliques of the same size). Unlike the outcome of Theorem 1, we require strongly sublinear separators; however, this stronger assumption holds in most natural examples of graph classes known to have sublinear separators. Also, such an assumption cannot be

avoided entirely: consider for example the class \mathcal{G} consisting of all graphs G such that the distance in G between any two vertices of degree at least 3 is at least $\log |V(G)|$. The class \mathcal{G} satisfies $s_{\mathcal{G}}(n) = O(n/\log n)$, but it has exponential expansion.

The major flaw in Theorem 2 is the assumption on bounded maximum degree, which severely restricts its applicability. While it is required in the proof, I have no reason to believe that it should be necessary and I propose the following conjecture.

Conjecture 1. *Every subgraph-closed class of graphs with strongly sublinear separators has subexponential expansion.*

The proof of Theorem 2 proceeds by contradiction, showing that a class with (nearly) exponentially large expansion cannot have strongly sublinear separators. The proof has two main ingredients. Firstly, it is relatively easy to deal with the situation when for arbitrarily large n , \mathcal{G} contains an n -vertex graph G such that $\nabla_{\log n}(G) \geq n^\varepsilon$ for some $\varepsilon > 0$, and to show that G has a subgraph without sufficiently small balanced separation. This is based on (a generalization of) the following theorem on shallow clique minors, which is of a separate interest in the context of previous results on the topic [27, 23].

Theorem 3. *For every ε such that $0 < \varepsilon \leq 1$, there exist integers $n_0, d \geq 0$ such that if a graph G on $n \geq n_0$ vertices has at least $n^{1+\varepsilon}$ edges, then it contains $K_{\lfloor n^{\varepsilon/6} \rfloor}$ as a d -minor.*

A variant of Theorem 3 was proved in my dissertation thesis [11], and we give a somewhat simplified version of the proof in Section 6. This part of the argument does not require bounded maximum degree.

It remains to consider the case that $\nabla_{\log n}(G) \ll n^\varepsilon$, and thus for a large $k \geq 0$, the graph G showing that $\nabla_k(\mathcal{G})$ is nearly exponential in k has many vertices (compared to any exponential in k). We would like to split G to components whose size makes it possible to apply the result of the previous paragraph, by removing a small part of G . Lipton and Tarjan [29] show that if \mathcal{G} has strongly sublinear separators, then we can remove some set S of vertices of G of sublinear size so that each component of $G - S$ has bounded size. However, we cannot directly apply this result, since some important part of the subgraph of G determining $\nabla_k(G)$ could be contained in S . Hence, as the second main ingredient, we need a strengthening of the result, showing that there exist many possible “almost disjoint” choices for S .

Lemma 4. *Let \mathcal{G} be a subgraph-closed class of graphs, with bounded maximum degree and strongly sublinear separators. There exists $b > 1$ with the following property. For every $\varepsilon > 0$ and every $G \in \mathcal{G}$, there exists some $m \geq 0$ and (not necessarily distinct) sets $S_1, \dots, S_m \subseteq V(G)$ such that each vertex of G is contained in at most εm of these sets and such that for $1 \leq i \leq m$, each component of $G - S_i$ has at most $b^{1/\varepsilon}$ vertices.*

The proof of Lemma 4 is given in Section 5, and the results are combined to a proof of Theorem 2 in Section 7. The notion of existence of a large number of almost disjoint subsets whose removal ensures some property can be viewed as a fractional version of a certain previously studied concept; we establish this connection in Section 2. The notion may be of separate interest because of its potential algorithmic applications. Although tangential to the topic of the paper, we give more details in Section 3.

1 Probabilistic results

We will use several tools from the probability theory, which we quickly recall here; for a more in-depth treatment see e.g. [30].

A *finite probability space* is a finite set S together with a *probability distribution* $\tau : S \rightarrow [0, 1]$ such that $\sum_{s \in S} \tau(s) = 1$. An *event* T is a subset of S , and its probability $\text{Prob}(T)$ is $\sum_{s \in T} \tau(s)$. For a unary predicate φ , we write $\text{Prob}[\varphi]$ as a shortcut for $\text{Prob}(\{s \in S : \varphi(s)\})$. A *random variable* is any function $X : S \rightarrow \mathbf{R}$, and its *expected value* is $E(X) = \sum_{s \in S} \tau(s)X(s)$.

We use several basic inequalities, such as Markov's inequality (see [30], Lemma 4.0.2).

Lemma 5. *Let X be a non-negative random variable. For any positive real number r ,*

$$\text{Prob}[X \geq r] \leq E(X)/r$$

and if $E(X) > 0$, then

$$\text{Prob}[X > r] < E(X)/r.$$

Random variables X_1, \dots, X_n are *independent* if for all measurable sets $A_1, \dots, A_n \subseteq \mathbf{R}$, we have

$$\text{Prob}[X_1 \in A_1, \dots, X_n \in A_n] = \text{Prob}[X_1 \in A_1] \text{Prob}[X_2 \in A_2] \dots \text{Prob}[X_n \in A_n].$$

We need the following corollary of Chernoff's bound (see [30], Theorem 7.2.1).

Lemma 6. *Let X_1, \dots, X_n be independent random variables, each of them attaining value 1 with probability p , and having value 0 otherwise. Let $X = X_1 + \dots + X_n$. Then*

$$\text{Prob}[X \leq np/2] < \exp\left(-\frac{3np}{28}\right).$$

Furthermore, for any real number $r \geq n$,

$$\text{Prob}[X \geq 2rp] < \exp\left(-\frac{3rp}{8}\right).$$

2 Fragility and fractional fragility

A *tree decomposition* (T, β) of a graph G is a tree T and a function $\beta : V(T) \rightarrow 2^{V(G)}$ assigning a *bag* $\beta(u) \subseteq V(G)$ to each vertex $u \in V(T)$, such that

- for every $v \in V(G)$, there exists $u \in V(T)$ with $v \in \beta(u)$,
- for every $vw \in E(G)$, there exists $u \in V(T)$ with $\{v, w\} \subseteq \beta(u)$, and
- for every $v \in V(G)$, the set $\{u : v \in \beta(u)\}$ induces a connected subtree of T .

The *width* of the decomposition is the maximum of the sizes of its bags minus one, and the treewidth of G is the minimum of the widths of its tree decompositions.

Consider a connected planar graph G . For each vertex $v \in V(G)$ and an integer $r > 0$, the subgraph of G induced by the vertices at distance at most r from v has treewidth at most $3r + 1$, as shown by Robertson and Seymour [38]. For integers $k \geq 1$ and $0 \leq t \leq k - 1$, let $Z_{t,k}$ denote the set of all vertices whose distance from v is congruent to t modulo k . As a corollary of the preceding observation, the graph $G - Z_{t,k}$ has treewidth at most $3k + 1$, see [16] for more details. This observation is very useful in the design of approximation algorithms, as for any set $X \subseteq V(G)$ (e.g., an optimal solution to an optimization problem), there exists t such that $|Z_{t,k} \cap X| \leq |X|/k$. Thus, it may be possible to find an optimal solution to a problem in $G - Z_{k,t}$ using its bounded treewidth, then extend it to a near-optimal solution in G . See Baker [1] for several algorithms along these lines.

Of course, instead of bounded treewidth, we could require any other property useful for the design of algorithms. This motivates the following definitions. A *class property* \mathcal{P} is a class of graph classes. For instance,

- let Tw denote the class property consisting of all graph classes with bounded treewidth;
- let Be denote the class property consisting of all classes with bounded expansion; and,
- let Vs denote the class property consisting of all graph classes with bounded component size, where a class \mathcal{C} has *bounded component size* if there exists some $t \geq 0$ such that every connected component of a graph in \mathcal{C} has at most t vertices.

Note that $\text{Vs} \subset \text{Tw} \subset \text{Be}$.

Let G be a graph and \mathcal{C} a class of graphs. A *packing* in G is a multiset of pairwise vertex-disjoint subsets of G (and thus only the empty set can appear multiple times in the packing). A packing P in G is \mathcal{C} -*complementary* if for every $X \in P$, the graph $G - X$ belongs to \mathcal{C} . A class of graphs \mathcal{G} is \mathcal{P} -*fragile* if for every $k \geq 1$, there exists a class $\mathcal{C} \in \mathcal{P}$ such that every graph in \mathcal{G} has a \mathcal{C} -complementary packing of size k (let us remark that the choice of \mathcal{C} is not necessarily unique). For a given integer $k \geq 1$, we say that a class \mathcal{C} with this property is a $(1/k)$ -*witness of the \mathcal{P} -fragility of \mathcal{G}* . We use $1/k$ rather than k for consistency with a notation we will introduce in a few paragraphs.

As we already outlined, the most studied version of fragility deals with treewidth, and the example we started with can be stated as the claim that the class of all planar graphs is Tw -fragile. One of the most general results in the area is by DeVos et al. [8], showing that every proper minor-closed class of graphs is Tw -fragile. Let us also remark similar concepts based on edge removal [8] or edge contraction [7]. Despite these encouraging results, it turns out that even very simple and well-structured classes of graphs need not be Tw -fragile.

Let R_n be the strong product of three paths with n vertices, that is, the graph with vertex set $\{(i, j, k) : 1 \leq i, j, k \leq n\}$ such that two distinct vertices (i_1, j_1, k_1) and (i_2, j_2, k_2) are adjacent iff $|i_1 - i_2| \leq 1$, $|j_1 - j_2| \leq 1$ and $|k_1 - k_2| \leq 1$; the graph R_4 is depicted in Figure 1 (the thickness of edges is just to aid the visualization). Let $\mathcal{R} = \{R_n : n \geq 1\}$.

Berger et al. [3] proved that for every $k \geq 0$, there exists $n \geq 1$ such that for every partition A, B of the vertices of R_n , either $R_n[A]$ or $R_n[B]$ has treewidth at least k . Hence, we get the following.

Theorem 7 (Berger et al. [3]). *The class \mathcal{R} is not Tw -fragile.*

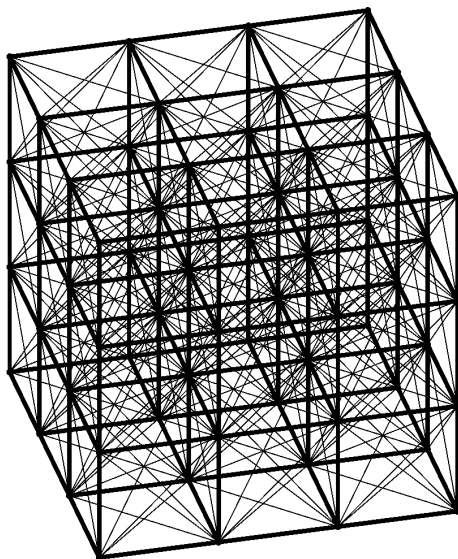


Figure 1: The graph R_4 .

This is problematic in our intended application, since the class \mathcal{R} has strongly sublinear separators. To overcome this issue, we introduce a fractional relaxation of fragility. Given a graph G and a class \mathcal{C} , let $G - \mathcal{C} = \{X \subseteq V(G) : G - X \in \mathcal{C}\}$. A *fractional \mathcal{C} -complementary packing* is an assignment $\pi : G - \mathcal{C} \rightarrow [0, 1]$ such that $\sum_{X \in G - \mathcal{C}} \pi(X) = 1$. The *thickness* of the fractional packing is

$$\max \left\{ \sum_{X \in G - \mathcal{C}, v \in X} \pi(X) : v \in V(G) \right\}.$$

Let us remark that if $G \in \mathcal{C}$, then G has a fractional \mathcal{C} -complementary packing of thickness 0 obtained by setting $\pi(\emptyset) = 1$ and $\pi(X) = 0$ for every nonempty $X \subseteq V(G)$. A convenient way how to view a fractional \mathcal{C} -complementary packing of thickness ε is as a probability distribution on $G - \mathcal{C}$ such that for every vertex v , the probability that v belongs to a set chosen at random according to this distribution is at most ε .

A class of graphs \mathcal{G} is *fractionally \mathcal{P} -fragile* if for every $\varepsilon > 0$, there exists a class $\mathcal{C} \in \mathcal{P}$ such that each graph in \mathcal{G} has a fractional \mathcal{C} -complementary packing of thickness at most ε . For a given $\varepsilon > 0$, we say that such a class \mathcal{C} is an ε -*witness of the fractional \mathcal{P} -fragility of \mathcal{G}* .

Clearly, if a class is \mathcal{P} -fragile, it is also fractionally \mathcal{P} -fragile. On the other hand, the following example together with Theorem 7 shows that fractional Tw-fragility (or even fractional Vs-fragility) does not imply Tw-fragility.

Lemma 8. *The class \mathcal{R} is fractionally Vs-fragile.*

Proof. Consider any $\varepsilon > 0$, and let $u = \lceil 3/\varepsilon \rceil$. Let $\mathcal{C} \in \text{Vs}$ be the class of graphs in that every component has at most $(u-1)^3$ vertices.

Consider a graph $R_n \in \mathcal{R}$. For $0 \leq t \leq u-1$, let $X_{n,t,u}$ denote the set of triples (i, j, k) such that $1 \leq i, j, k \leq n$ and at least one of i, j and k is congruent to t modulo u . Then each component of $R_n \setminus X_{n,t,u}$ has at most $(u-1)^3$ vertices, and thus $X_{n,t,u}$ belongs to $R_n - \mathcal{C}$. If $n \leq u-1$, then set $\pi(\emptyset) = 1$ and $\pi(X) = 0$ for every non-empty $X \in R_n - \mathcal{C}$. If $n \geq u$, then set $\pi(X_{n,t,u}) = 1/u$ for $0 \leq t \leq u-1$ and $\pi(X) = 0$ for every other $X \in R_n - \mathcal{C}$.

Note that every vertex v of R_n belongs to at most three of the sets $X_{n,t,u}$, and thus the probability that v belongs to a set chosen according to the described distribution is at most $3/u \leq \varepsilon$. Thus, π is a fractional \mathcal{C} -complementary packing in R_n of thickness at most ε .

Since such a fractional packing exists for every $\varepsilon > 0$ and $R_n \in \mathcal{R}$, it follows that \mathcal{R} is fractionally Vs-fragile. \square

Let us remark that it is not a coincidence that \mathcal{R} is not only fractionally Tw-fragile but also fractionally Vs-fragile, as we will see in Corollary 20.

3 Properties and applications of fractional fragility

As we have already mentioned in the introduction, we need the notion of fractional Vs-fragility when showing the subexponential expansion property of graph classes with strongly sublinear separators. Nevertheless, the notion of fractional \mathcal{P} -fragility appears to be of independent interest. We can consider it to be a measure of the distance of the graph class from some property. Also, many of the algorithmic applications of Tw-fragility also work for the fractional relaxation, which extends them to more graph classes.

This section is devoted to establishing the basic properties of fractional fragility and showcasing some of its applications. While this may help the reader to obtain a better understanding of the notion, we do not use these results in the rest of the paper, and thus the reader may skip to the next section if they prefer to.

Let us first give two examples of algorithmic applications of fractional Tw-fragility (both of which are straightforward generalizations of previously

known results for Tw-fragility). Of course, in this context we need to be able to find the fractional packings that certify the fractional Tw-fragility efficiently. For $c \geq 1$, we say that a class \mathcal{G} of graphs is $O(n^c)$ -effectively fractionally Tw-fragile if for every integer $k \geq 1$, there exists a constant p_k , a $(1/k)$ -witness \mathcal{C}_k of the fractional Tw-fragility of \mathcal{G} , and an algorithm with

input: a graph $G \in \mathcal{G}$, and

output: a fractional \mathcal{C}_k -complementary packing of thickness at most $1/k$ in G , which assigns a non-zero value to at most p_k elements of $G - \mathcal{C}_k$,

and the time complexity of the algorithm is $O(|V(G)|^c)$.

The *independence number* $\alpha(G)$ is the size of the largest independent set of a graph G . Determining the independence number of a graph is an NP-complete problem [20], and even approximating it up to a polynomial factor is not possible in polynomial time unless $P = NP$ [2]. Nevertheless, a polynomial-time approximation scheme for the independent set exists for graphs from any $O(n^c)$ -effectively fractionally Tw-fragile class of graphs.

Lemma 9. *Let $c \geq 1$ and let \mathcal{G} be an $O(n^c)$ -effectively fractionally Tw-fragile class of graphs. For every $\varepsilon > 0$, there exists an algorithm with time complexity $O(|V(G)|^c)$ that for a graph $G \in \mathcal{G}$ returns an independent set of G of size at least $(1 - \varepsilon)\alpha(G)$.*

Proof. Let $k = \lceil 1/\varepsilon \rceil$. The algorithm first finds a fractional \mathcal{C}_k -complementary packing π in G of thickness at most $1/k \leq \varepsilon$, using the algorithm from the definition of $O(n^c)$ -effective fractional Tw-fragility. Let X_1, \dots, X_p be the elements of $G - \mathcal{C}_k$ to which π assigns a non-zero probability, where $p \leq p_k$. For $1 \leq i \leq p$, let A_i be a largest independent set in $G - X_i$, which can be found in linear time since \mathcal{C}_k has bounded treewidth [5]. The algorithm returns the largest of A_1, \dots, A_p .

Let A be a largest independent set in G . For a set $X \in G - \mathcal{C}_k$ chosen at random according to the probability distribution π , each vertex belongs to X with probability at most $1/k \leq \varepsilon$, and thus the expected size of $X \cap A$ is at most $\varepsilon|A|$. Hence, there exists $X \in G - \mathcal{C}_k$ with $\pi(X) > 0$ such that $|X \cap A| \leq \varepsilon|A|$. Since $A \setminus X$ is an independent set in $G - X$, we conclude that $\alpha(G - X) \geq |A \setminus X| \geq (1 - \varepsilon)|A|$. Therefore, the algorithm indeed returns an independent set of size at least $(1 - \varepsilon)\alpha(G)$. \square

Another problem that we consider is testing the existence of a subgraph. Testing whether a clique K_n is a subgraph of G is equivalent to verifying that the complement of G has independence number at least n , and

thus if the tested subgraph is a part of the input, then the problem is NP-complete. To test whether a fixed graph H is a subgraph of G , we can test all $O(|V(G)|^{|V(H)|})$ choices for the possible placement of the vertices of H in G , or we can use a more involved algorithm of Nešetřil and Poljak [36]. In both cases, we obtain a polynomial-time algorithm whose exponent depends on H , and this cannot be avoided in general unless $\text{FPT} = \text{W}[1]$, see [9]. However, if G is furthermore restricted to belong to an $O(n^c)$ -effectively fractionally Tw-fragile class of graphs, we can design a polynomial-time algorithm whose exponent is independent of H .

Lemma 10. *Let $c \geq 1$ and let H be a fixed graph. If a class \mathcal{G} is $O(n^c)$ -effectively fractionally Tw-fragile, then there exists an algorithm determining whether $H \subseteq G$ for graphs $G \in \mathcal{G}$ with time complexity $O(|V(G)|^c)$.*

Proof. Let $k = |V(H)| + 1$. The algorithm finds a \mathcal{C}_k -complementary packing π in G of thickness at most $1/k$, using the algorithm from the definition of $O(n^c)$ -effective fractional Tw-fragility. Let X_1, \dots, X_p be the elements of $G - \mathcal{C}_k$ to which π assigns a non-zero probability, where $p \leq p_k$. For $1 \leq i \leq p$, determine whether $H \subseteq G - X_i$ in linear time, since \mathcal{C} has bounded treewidth. If H is a subgraph of one of $G - X_1, \dots, G - X_n$, then H is also a subgraph of G . Otherwise, the algorithm returns that H is not a subgraph of G .

Clearly, if H is not a subgraph G , then the algorithm correctly determines this. Suppose that H is a subgraph of G , and let $S \subseteq V(G)$ be the set of vertices of this subgraph. For a set $X \in G - \mathcal{C}_k$ chosen at random according to the probability distribution π , the expected size of $X \cap S$ is at most $|S|/k < 1$. Hence, there exists $X \in G - \mathcal{C}_k$ with $\pi(X) > 0$ such that $X \cap S = \emptyset$, and thus $H \subseteq G - X$. It follows that the algorithm correctly determines whether $H \subseteq G$. \square

In these algorithms, bounded treewidth could be replaced by any other class property which ensures efficient solvability of the considered problem. Furthermore, the notion of efficiency could be relaxed, and we could for instance only require to be able to sample from the probability distribution efficiently (which would turn the algorithms to probabilistic ones).

In the rest of the text, we do not consider the algorithmic constraint of being able to find the packings efficiently, and only discuss the graph-theoretic questions concerning fragility and fractional fragility (although, let us remark that the argument proving Lemma 4 can be implemented in an $O(n^c)$ -effective way for every $c > 2$). The following is obvious.

Observation 11. *Let \mathcal{G}_1 and \mathcal{G}_2 be graph classes, and let \mathcal{P}_1 and \mathcal{P}_2 be class properties.*

- *If \mathcal{G}_1 is \mathcal{P}_1 -fragile, it is also fractionally \mathcal{P}_1 -fragile.*
- *If \mathcal{G}_1 is (fractionally) \mathcal{P}_1 -fragile and $\mathcal{G}_2 \subseteq \mathcal{G}_1$, then \mathcal{G}_2 is (fractionally) \mathcal{P}_1 -fragile.*
- *If $\mathcal{P}_1 \subseteq \mathcal{P}_2$, and \mathcal{G}_1 is (fractionally) \mathcal{P}_1 -fragile, then \mathcal{G}_1 is (fractionally) \mathcal{P}_2 -fragile.*

Fractional fragility is transitive in the following sense (so, for example, if a class \mathcal{G} is fractionally \mathcal{P} -fragile for a class property \mathcal{P} whose elements contain only planar graphs, then \mathcal{G} is also fractionally Tw-fragile).

Lemma 12. *Let \mathcal{P}_1 and \mathcal{P}_2 be class properties such that every class in \mathcal{P}_1 is fractionally \mathcal{P}_2 -fragile. If a class \mathcal{G} is fractionally \mathcal{P}_1 -fragile, then it also is fractionally \mathcal{P}_2 -fragile.*

Proof. Consider any $\varepsilon > 0$. Let \mathcal{C}_1 be a $(\varepsilon/2)$ -witness of the fractional \mathcal{P}_1 -fragility of \mathcal{G} . Since $\mathcal{C}_1 \in \mathcal{P}_1$, the class \mathcal{C}_1 is fractionally \mathcal{P}_2 -fragile. Let \mathcal{C}_2 be an $(\varepsilon/2)$ -witness of the fractional \mathcal{P}_2 -fragility of \mathcal{C}_1 .

Consider a graph $G \in \mathcal{G}$ and let π_1 be its fractional \mathcal{C}_1 -complementary packing of thickness at most $\varepsilon/2$. For every $Z \in G - \mathcal{C}_1$, let π_Z be a fractional \mathcal{C}_2 -complementary packing of thickness at most $\varepsilon/2$ of $G - Z$. Let $X \in G - \mathcal{C}_2$ be chosen at random as follows: First, select $X_1 \in G - \mathcal{C}_1$ at random according to the distribution π_1 . Then, select $X_2 \in (G - X_1) - \mathcal{C}_2$ at random according to the distribution π_{X_1} . Let $X = X_1 \cup X_2$. This procedure for choosing $X \in G - \mathcal{C}_2$ defines a probability distribution π on $G - \mathcal{C}_2$. The probability that a vertex $v \in V(G)$ belongs to X chosen at random according to the distribution π is equal to the probability that either $v \in X_1$, or $v \notin X_1$ and $v \in X_2$. Each of these probabilities is at most $\varepsilon/2$, and thus the probability that v belongs to X is at most ε . Therefore, π is a fractional \mathcal{C}_2 -complementary packing in G of thickness at most ε .

Since such a fractional \mathcal{C}_2 -complementary packing exists for every $G \in \mathcal{G}$, it follows that \mathcal{C}_2 is an ε -witness of fractional \mathcal{P}_2 -fragility of \mathcal{G} . As the choice of $\varepsilon > 0$ was arbitrary, we conclude that \mathcal{G} is fractionally \mathcal{P}_2 -fragile. \square

4 Fragility and bounded expansion

Let us now derive the connection to bounded expansion, which we use in the proof of Theorem 2. Let us recall that Be denotes the class property

consisting of all classes with bounded expansion. For a function $f : \mathbf{N} \rightarrow \mathbf{N}$, let $\text{Ex}(f)$ denote the class of all graphs G such that $\nabla_k(G) \leq f(k)$ for all integers $k \geq 0$. Note that for every fractionally Be-fragile class of graphs \mathcal{G} , there exists a function $g : \mathbf{R}^+ \times \mathbf{N} \rightarrow \mathbf{N}$ such that for every $\varepsilon > 0$, the class $\text{Ex}(g(\varepsilon, \cdot))$ is an ε -witness of the fractional Be-fragility of \mathcal{G} . If g satisfies this property, we say that \mathcal{G} is *fractionally (Be, g)-fragile*.

As the following lemma shows, a class of graphs is fractionally Be-fragile if and only if it has bounded expansion (and consequently, if and only if it is Be-fragile), and thus the notion of the fractional Be-fragility does not bring anything qualitatively new. Nevertheless, the quantitative relationship between the respective expansion functions will be of importance later.

Lemma 13. *Let $g : \mathbf{R}^+ \times \mathbf{N} \rightarrow \mathbf{N}$ be an arbitrary function, and let us define a function $f : \mathbf{N} \rightarrow \mathbf{N}$ by setting $f(k) = 2g\left(\frac{1}{4k+4}, k\right)$. If \mathcal{G} is a fractionally (Be, g)-fragile class of graphs, then the expansion of \mathcal{G} is bounded by f .*

Proof. Fix $k \geq 0$. Let $\varepsilon = \frac{1}{4k+4}$ and let $\mathcal{C} = \text{Ex}(g(\varepsilon, \cdot))$.

Consider an arbitrary graph $G \in \mathcal{G}$ and let H be a k -minor of G . Let $V(H) = \{v_1, \dots, v_h\}$. The presence of H as a k -minor of G is certified by vertex-disjoint rooted trees $T_1, \dots, T_h \subseteq G$ such that

- for $1 \leq i \leq h$, the tree T_i has depth at most k , and
- if $v_i v_j \in E(H)$, then there exists an edge $e_{ij} \in E(G)$ joining a vertex of T_i with a vertex of T_j .

For an edge $v_i v_j \in E(H)$, let P_{ij} be the path of length at most $2k + 1$ consisting of e_{ij} and the paths from the ends of e_{ij} to the roots of T_i and T_j .

Since \mathcal{G} is fractionally (Be, g)-fragile, there exists a fractional \mathcal{C} -complementary packing π in G of thickness at most ε . Let $X \in G - \mathcal{C}$ be chosen at random according to π . Let H' be the subgraph of H consisting of edges $v_i v_j$ such that P_{ij} is disjoint with X , and of the vertices incident with these edges. The probability that P_{ij} intersects X is at most $\varepsilon |V(P_{ij})| \leq 1/2$, and thus the expected number of edges of H' is at least $|E(H)|/2$. Let us fix a set $X \in G - \mathcal{C}$ so that $|E(H')| \geq |E(H)|/2$. Note that H' is a k -minor of $G - X$, and thus $\frac{|E(H')|}{|V(H')|} \leq g(\varepsilon, k)$. However, $\frac{|E(H')|}{|V(H')|} \geq \frac{|E(H)|/2}{|V(H')|} \geq \frac{1}{2} \cdot \frac{|E(H)|}{|V(H)|}$.

It follows that $\frac{|E(H)|}{|V(H)|} \leq 2g(\varepsilon, k) = f(k)$ for every k -minor H of G , and thus $\nabla_k(G) \leq f(k)$. Since this holds for every $G \in \mathcal{G}$ and every $k \geq 0$, the expansion of \mathcal{G} is bounded by f . \square

5 Fragility and sublinear separators

Next, we study the connection between fractional fragility and sublinear separators.

Lemma 14. *Let \mathcal{P} be a class property such that every class in \mathcal{P} has sublinear separators. If \mathcal{G} is a fractionally \mathcal{P} -fragile class of graphs, then \mathcal{G} has sublinear separators.*

Proof. Suppose for a contradiction that $\limsup_{n \rightarrow \infty} s_{\mathcal{G}}(n)/n = \delta > 0$. Let \mathcal{C} be a $(\delta/4)$ -witness of the fractional \mathcal{P} -fragility of \mathcal{G} ; by the assumptions, we have $\lim_{n \rightarrow \infty} s_{\mathcal{C}}(n)/n = 0$. Let $n_0 \geq 0$ be the smallest integer such that $s_{\mathcal{C}}(n) < \frac{\delta}{4}n$ for every $n \geq n_0$.

Since $\limsup_{n \rightarrow \infty} s_{\mathcal{G}}(n)/n = \delta$, there exists a graph $G \in \mathcal{G}$ on $n \geq n_0$ vertices such that every balanced separation in G has size at least $\frac{\delta}{2}n$. Let π be a fractional \mathcal{C} -complementary packing of thickness at most $\delta/4$ in G . The expected size of a set chosen at random according to π is at most $\delta n/4$, and thus there exists $X \in G - \mathcal{C}$ such that $|X| \leq \delta n/4$. Let (A', B') be a balanced separation of $G - X$ of size at most $s_{\mathcal{C}}(n) < \frac{\delta}{4}n$. Let (A, B) be a corresponding separation of G with $A' \subseteq A$, $B' \subseteq B$ and $V(A) \cap V(B) = (V(A') \cap V(B')) \cup X$. Note that (A, B) is a balanced separation in G . However, $|V(A) \cap V(B)| \leq |X| + s_{\mathcal{C}}(n) < \frac{\delta}{2}n$, which contradicts the choice of G . \square

By the previous lemma, having sublinear separators is a necessary condition for fractional Vs-fragility. Further necessary condition is bounded maximum degree.

Observation 15. *If a class \mathcal{G} is fractionally Vs-fragile, then it has bounded maximum degree.*

Proof. Let \mathcal{C} be a $(1/3)$ -witness of the fractional Vs-fragility of \mathcal{G} . Let s be the maximum size of a component of a graph from \mathcal{C} .

Consider any graph $G \in \mathcal{G}$, and let π be its fractional \mathcal{C} -complementary packing of thickness at most $1/3$. Let v be any vertex of G and let N be the set of all neighbors of v . The probability that a set X chosen at random according to π contains v is at most $1/3$. Furthermore, the expected size of the intersection of X and N is at most $|N|/3$, and by Lemma 5, the probability that $|X \cap N| > |N|/2$ is less than $2/3$. Therefore, there exists $X \in G - \mathcal{C}$ such that $v \notin X$ and $|X \cap N| \leq |N|/2$. Since $G - X$ contains v and at least $|N|/2$ of its neighbors, it has a component of size at least $|N|/2 + 1$. Since every component of $G - X$ has size at most s , it follows

that $|N| \leq 2s - 2$. Therefore, every graph from \mathcal{G} has maximum degree at most $2s - 2$. \square

As the main result of this section, we show that for subgraph-closed graph classes, these necessary conditions are almost sufficient (we require strongly sublinear separators). Let us recall a strong separation property for graphs with bounded tree-width, see [39].

Lemma 16. *For any graph G and a set $X \subseteq V(G)$, there exists a separation (A, B) of G of size at most $\text{tw}(G) + 1$ such that $|X \setminus V(A)|, |X \setminus V(B)| \leq 2|X|/3$.*

We need an auxiliary lemma.

Lemma 17. *Every graph G has a rooted tree decomposition (T, β) with bags of size at most $12(\text{tw}(G) + 1)(\Delta(G) + 1)$ such that every vertex of T has at most two sons and for each $v \in V(G)$, the subtree $T[\{u : v \in \beta(u)\}]$ has depth at most $1 + 4\log(\Delta(G) + 1)$.*

Proof. Let $b = 12(\text{tw}(G) + 1)$ and $w = \Delta(G)b$.

Let us consider the following algorithm to obtain a tree decomposition.

input: A graph H of maximum degree at most $\Delta(G)$ and tree-width at most $\text{tw}(G)$, and a set $Z \subseteq V(H)$ of size at most w (which we call the *root set*).

output: A rooted tree decomposition of H with bags of size at most $w + b$, whose root bag contains Z .

- If $|V(H)| \leq w + b$, then let the decomposition consist of a single bag containing all vertices.
- If $|Z| \leq b$, then let Z' be a superset of Z of size b and let Z'' consist of all vertices of $V(H) \setminus Z'$ that have a neighbor in Z' . Let us apply the algorithm recursively to $H - Z'$ with the root set Z'' (note that this is possible, since $|Z''| \leq \Delta(H)|Z'| \leq w$). To the root of the resulting decomposition, attach a father node whose bag is $Z' \cup Z''$.
- Otherwise, let (A, B) be a separation of H of size at most $\text{tw}(H) + 1$ with $|Z \setminus V(A)|, |Z \setminus V(B)| \leq 2|Z|/3$ which exists by Lemma 16. Let $Z_A = (V(A) \cap V(B)) \cup (Z \setminus V(B))$ and $Z_B = (V(A) \cap V(B)) \cup (Z \setminus V(A))$. Apply the algorithm recursively to A with the root set Z_A , and to B with the root set Z_B (this is possible, since $|Z_A|, |Z_B| < |Z| \leq w$ as we argue below). Add a common father node of their roots with bag $Z \cup V(A \cap B)$ to the resulting tree decomposition.

To obtain the required tree decomposition of G , we run the described algorithm for G with the empty root set. Note that the case that $|Z| > b$ can only be reached if $\Delta(G) \geq 2$, and that in this case $|Z_A|, |Z_B| \leq 2|Z|/3 + \text{tw}(G) + 1 < 3|Z|/4$. Therefore, after at most $\lceil \log(w/b)/\log(4/3) \rceil \leq 4 \log(\Delta(G) + 1)$ levels of recursion, we reach the case that $|Z| \leq b$, and all the vertices of Z are excluded from the graph in the next recursive call. Hence, every vertex appears in the bags of at most $2 + 4 \log(\Delta(G) + 1)$ consecutive levels of the tree decomposition. \square

It is easy to see that if every subgraph of an n -vertex graph G has a balanced separator of size at most b , then G has treewidth $O(b \log n)$. Recently, a stronger claim was proved (the weaker bound with the logarithmic factor would suffice for the purposes of this paper, however using Theorem 18 simplifies the computations a bit).

Theorem 18 (Dvořák and Norin [14]). *If \mathcal{G} is a subgraph-closed class of graphs, then every graph $G \in \mathcal{G}$ has treewidth at most $105s_{\mathcal{G}}(|V(G)|)$.*

We are now ready to prove the main result of this section, which is a reformulation of Lemma 4.

Lemma 19. *Let \mathcal{G} be subgraph-closed class of graphs. If \mathcal{G} has bounded maximum degree and strongly sublinear separators, then \mathcal{G} is fractionally Vs -fragile. Furthermore, there exists a constant $b > 1$ such that for every $0 < \varepsilon \leq 1$, the class \mathcal{C}_ε of all graphs in \mathcal{G} such that all their components have at most $b^{1/\varepsilon}$ vertices is an ε -witness of the fractional Vs -fragility of \mathcal{G} .*

Proof. Let $c \geq 1$ and $0 \leq \delta < 1$ be real numbers such that $s_{\mathcal{G}}(n) \leq cn^\delta$ for every $n \geq 0$. Let $\Delta \geq 0$ be an integer such that every graph in \mathcal{G} has maximum degree at most Δ . Let $\iota > 0$ be chosen arbitrarily so that $\delta + \iota < 1$. Let $c_1 = 105c$, so that every graph in \mathcal{G} on n vertices has treewidth at most $c_1 n^\delta$ by Theorem 18. Let $c_2 = 24(c_1 + 1)(\Delta + 1)$, $c'_2 = c_2^{\frac{1}{1-(\delta+\iota)}}$, $c_3 = \frac{1}{\delta+\iota} > 1$, $c_4 = \frac{2+4\log(\Delta+1)}{(c_3-1)\iota}$, $c_5 = e^{c_3 c_4}$ and $b = c'_2 c_5$.

We first prove the following auxiliary claim:

(\star) *Let G be a graph in \mathcal{G} with at most n vertices, and let*

$$S = \{X \subseteq V(G) : \text{every component of } G - X \text{ has at most } c_2 n^{\delta+\iota} \text{ vertices}\}.$$

There exists a probability distribution $\pi_{G,n}$ on S such that every vertex of G has probability at most

$$\frac{2 + 4 \log(\Delta + 1)}{\iota \log n}$$

of appearing in a set chosen according to this distribution.

By the choice of c_1 , the graph G has tree-width at most $c_1 n^\delta$. Let (T, β) be the rooted tree decomposition of G obtained using Lemma 17. Recall that each vertex of T has at most two sons, and note that each bag of the decomposition has size at most $12(c_1 n^\delta + 1)(\Delta + 1) \leq 12(c_1 + 1)(\Delta + 1)n^\delta = \frac{c_2}{2}n^\delta$.

Let $k = \lceil \iota \log n \rceil$ and for $0 \leq i \leq k - 1$, let X_i be the set of vertices of G which appear in the bags of the decomposition whose distance d from the root satisfies $d \equiv i \pmod{k}$. Consider any connected component H of $G - X_i$. Let T_H be the subtree of T induced by $\{u : \beta(u) \cap V(H) \neq \emptyset\}$ and let $\beta_H : V(T_H) \rightarrow 2^{V(H)}$ be defined by $\beta_H(u) = \beta(u) \cap V(H)$ for $u \in V(T_H)$. Then (T_H, β_H) is a rooted tree decomposition of H such that each vertex of T_H has at most two sons and each bag of T_H has size at most $\frac{c_2}{2}n^\delta$. Furthermore, by the choice of X_i , the tree T_H has depth at most $k - 2$, hence $|V(T_H)| \leq 2^k \leq 2^{\iota \log n + 1} \leq 2n^\iota$ and $|V(H)| \leq \frac{c_2}{2}n^\delta |V(T_H)| \leq c_2 n^{\delta + \iota}$. Since this holds for every connected component of $G - X_i$, the set X_i belongs to S .

For every $X \in S$, let $\pi_{G,n}(X) = \frac{|\{i : 0 \leq i \leq k-1, X_i = X\}|}{k}$. Since we chose (T, β) using Lemma 17, each vertex of G appears in at most $2 + 4 \log(\Delta + 1)$ of the sets X_0, \dots, X_{k-1} , and thus the probability that a vertex appears in a set chosen according to the distribution $\pi_{G,n}$ is at most $\frac{2+4 \log(\Delta+1)}{k} \leq \frac{2+4 \log(\Delta+1)}{\iota \log n}$. This finishes the proof of (\star) .

We now iterate this construction for an n -vertex graph $G \in \mathcal{G}$. Let $n_0 = n$ and $n_{i+1} = c_2 n_i^{\delta + \iota}$ for $i \geq 0$. Note that

$$n_i \leq c_2^{1+(\delta+\iota)+(\delta+\iota)^2+\dots} n^{(\delta+\iota)^i} = c_2^{\frac{1}{1-(\delta+\iota)}} n^{(\delta+\iota)^i} = c'_2 n^{(\delta+\iota)^i} = c'_2 n^{1/c_3^i} \quad (1)$$

and since $c_2 \geq 1$,

$$n_i \geq n^{(\delta+\iota)^i} = n^{1/c_3^i}. \quad (2)$$

Consider a graph $G \in \mathcal{G}$ with n vertices. Let $G_0 = G$. For $i \geq 0$, G_i will be some subgraph of G such that each component of G_i has size at most n_i . Note that since \mathcal{G} is subgraph-closed, every component of G_i belongs to \mathcal{G} . To construct G_{i+1} , for each component G'_i of G_i consider the probability distribution $\pi_{G'_i, n_i}$ obtained in (\star) and choose a subset of $V(G'_i)$ at random according to this distribution (independently in each component). Let X_i be the union of all these subsets and let $G_{i+1} = G_i - X_i$.

For an integer $t \geq 0$, let $Y_t = X_0 \cup X_1 \cup \dots \cup X_{t-1}$. Note that Y_t is a subset of $V(G)$ chosen at random according to a probability distribution described

by the construction of the previous paragraph, and that each component of $G - Y_t$ has size at most n_t . The probability that a vertex $v \in V(G)$ belongs to this set Y_t is at most

$$\sum_{i=0}^{t-1} \text{Prob}[v \in X_i] \leq \sum_{i=0}^{t-1} \frac{2 + 4 \log(\Delta + 1)}{\iota \log n_i} = \frac{2 + 4 \log(\Delta + 1)}{\iota} \sum_{i=0}^{t-1} \frac{1}{\log n_i} \quad (3)$$

by (\star) . By (2) we have $\log n_i \geq \frac{1}{c_3^i} \log n$, and thus

$$\sum_{i=0}^{t-1} \frac{1}{\log n_i} \leq \frac{1}{\log n} \sum_{i=0}^{t-1} c_3^i = \frac{c_3^t - 1}{(c_3 - 1) \log n} \leq \frac{c_3^t}{(c_3 - 1) \log n}.$$

Combined with (3), this implies that the probability that v belongs to Y_t is at most

$$\frac{2 + 4 \log(\Delta + 1)}{\iota} \sum_{i=0}^{t-1} \frac{1}{\log n_i} \leq \frac{(2 + 4 \log(\Delta + 1)) c_3^t}{\iota (c_3 - 1) \log n} = \frac{c_4 c_3^t}{\log n}. \quad (4)$$

Consider any ε such that $0 < \varepsilon \leq 1$. Recall that \mathcal{C}_ε is the class of all graphs in \mathcal{G} such that all their components have at most $b^{1/\varepsilon}$ vertices.

If $\log n < c_4/\varepsilon$, then $|V(G)| = n = e^{\log n} < e^{c_4/\varepsilon} < b^{1/\varepsilon}$, and thus $G \in \mathcal{C}_\varepsilon$ and setting $\pi(\emptyset) = 1$ and $\pi(X) = 0$ for every non-empty $X \subseteq V(G)$ gives a fractional \mathcal{C}_ε -complementary packing in G of thickness 0.

Suppose now that $\log n \geq c_4/\varepsilon$. Fix $t \geq 0$ as the largest integer such that $c_4 c_3^t / \log n \leq \varepsilon$, and let π be the probability distribution on the subsets of $V(G)$ described by the process generating the set Y_t (so Y_t is chosen at random from the distribution π). By the maximality of t , we have $c_3^{t+1} > \frac{\varepsilon \log n}{c_4}$, and $c_3^t > \frac{\varepsilon \log n}{c_3 c_4}$. As we observed before, each component of $G - Y_t$ has size at most n_t . However, by (1),

$$n_t \leq c_2' n^{1/c_3^t} \leq c_2' n^{\frac{c_3 c_4}{\varepsilon \log n}} = c_2' e^{\frac{c_3 c_4}{\varepsilon}} = c_2' c_5^{1/\varepsilon} \leq b^{1/\varepsilon},$$

and thus $G - Y_t \in \mathcal{C}_\varepsilon$. By (4), the probability that a vertex $v \in V(G)$ belongs to Y_t is at most $\frac{c_4 c_3^t}{\log n} \leq \varepsilon$. Therefore, π is a fractional \mathcal{C}_ε -complementary packing in G of thickness at most ε .

Since this holds for every $\varepsilon > 0$, we conclude that \mathcal{G} is fractionally Vs-fragile and that \mathcal{C}_ε is an ε -witness of the fractional Vs-fragility of \mathcal{G} . \square

Let us remark that for every $k \geq 0$, the class of all graphs with treewidth at most k has strongly sublinear separators. Hence, Lemma 19 together with Lemma 12 implies the following.

Corollary 20. *Every fractionally Tw-fragile class with bounded maximum degree is fractionally Vs-fragile.*

6 Expansion in small graphs

Roughly, the aim of this section is to show that for $\varepsilon > 0$, if a sufficiently large n -vertex graph G satisfies $\nabla_{\lceil \log n \rceil}(G) \geq n^\varepsilon$, then G contains a subgraph without a strongly sublinear separation. To do so, we show that for some $\varepsilon' > 0$, G contains a shallow minor of a clique K_s with $s \geq n^{\varepsilon'}$. This clique contains a 3-regular expander on s vertices as a subgraph, and thus G contains a shallow subdivision of this 3-regular expander. It is easy to see that such a subdivision does not have strongly sublinear separators.

Consider a $\lceil \log n \rceil$ -minor G' of G with edge density $\nabla_{\lceil \log n \rceil}(G) \geq n^\varepsilon$. By Komlós and Szemerédi [26] and Thomasson [40], G' contains $K_{\lfloor n^{\varepsilon/2} \rfloor}$ as a minor (actually, a topological minor). However, their proofs give no bound on the depth of the minor. Topological minors with edges subdivided bounded number of times were studied by Kostochka and Pyber [27] and Jiang [23], however their results do not give polynomially large cliques. Hence, we need to derive a result combining both shallowness and polynomial size. Let us remark that doing so in the terms of topological minors is possible [11], however it will be more convenient to only give the result for minors.

We are going to repeatedly take shallow minors, and the following observation will be useful.

Observation 21. *If H' is a d_1 -minor of G and H is a d_2 -minor of H' , then H is a $(d_1 + d_2(2d_1 + 1))$ -minor of G .*

We also use the following result, which gives shallow minors in very dense graphs. Let K'_t denote the graph obtained from K_t by subdividing each edge by exactly one vertex.

Lemma 22 (Jiang [23, Proposition 2.3]). *For any $t \geq 1$, if a graph G on n vertices has at least $t^2 n^{3/2}$ edges, then G contains K'_t as a subgraph.*

For sparser graphs, we use the following lemma to find denser 1-minors.

Lemma 23. *Suppose that $c \geq 64$, $t \geq 1$ and $0 < \varepsilon < 1$. If a graph G on n vertices has at least $ct^4 n^{1+\varepsilon}$ edges, then it contains either a graph G' with at least $\frac{c}{32} t^4 |V(G')|^{1+\varepsilon+\varepsilon^2}$ edges as a 1-minor, or K_t as a 4-minor.*

Proof. Note that removing a vertex of degree at most $ct^4 n^\varepsilon$ from G results in a graph on $n - 1$ vertices and with at least $ct^4(n - 1)n^\varepsilon \geq ct^4(n - 1)^{1+\varepsilon}$ edges; hence, without loss of generality we can assume that the minimum degree of G is at least $ct^4 n^\varepsilon$. Consequently, we have

$$n \geq ct^4 n^\varepsilon \text{ and } n^{1-\varepsilon} \geq ct^4. \quad (5)$$

Let $A \subseteq V(G)$ be chosen so that the number of edges of G with exactly one end in A is as large as possible, let $B = V(G) \setminus V(A)$ and let G_1 be the spanning bipartite subgraph of G consisting of edges of G with exactly one end in A . Consider any vertex $v \in V(G)$, and let A_v be the symmetric difference of A and $\{v\}$. Observe that G contains at least $|E(G_1)| - \deg_{G_1}(v) + (\deg_G(v) - \deg_{G_1}(v))$ edges with exactly one end in A_v , and by the choice of A , it follows that $\deg_{G_1}(v) \geq \frac{1}{2} \deg_G(v)$. Hence, every vertex of G_1 has degree at least $\frac{c}{2} t^4 n^\varepsilon$. By symmetry, we can assume that $|A| \leq n/2 \leq |B|$.

Let $p = n^{-\varepsilon}$. Let A' be a subset of A obtained by choosing each vertex independently at random with probability p . Since $|A| \leq n/2$, Lemma 6 with $r = n/2$ and (5) implies that the probability that $|A'| \geq 2rp = n^{1-\varepsilon}$ is less than

$$\exp\left(-\frac{3n^{1-\varepsilon}}{16}\right) \leq \exp\left(-\frac{3ct^4}{16}\right) < 1/3. \quad (6)$$

Consider a vertex $v \in B$. The expected number of neighbors of B in A' is at least $\delta(G_1)p \geq \frac{c}{2} t^4$, and by Lemma 6, the probability that the number of neighbors of B in A' is less than $\frac{c}{4} t^4$ is at most

$$\exp\left(-\frac{3ct^4}{56}\right) < 1/3.$$

Let B_1 consist of the vertices of B with less than $\frac{c}{4} t^4$ neighbors in A' ; the expected size of B_1 is at most $|B|/3$. By Lemma 5, the probability that $|B_1| \geq |B|/2$ is at most $2/3$. Let $B' = B \setminus B_1$ consist of the vertices of B with at least $\frac{c}{4} t^4$ neighbors in A' ; hence,

$$\text{Prob}[|B'| \leq |B|/2] \leq 2/3. \quad (7)$$

By (6) and (7), we have $|A'| < n^{1-\varepsilon}$ and $|B'| > |B|/2 \geq n/4$ with non-zero probability; let us fix a set $A' \subset A$ of size less than $n^{1-\varepsilon}$ such that the set $B' \subseteq B$ of vertices of B with at least $\frac{c}{4} t^4$ neighbors in A' has size greater than $n/4$.

We now form a 1-minor G' of G_1 as follows. Let the vertex set of G' be A' and initially, let the edge set of G' be empty. We process each vertex $v \in B'$ in turn. Let N_v be the set of neighbors of v in A' . If $G'[N_v]$ has a vertex w of degree at most $|N_v|/2 - 1$, then we add edges between w and all other vertices of N_v to G' (this corresponds to contracting the edge vw of G_1) and continue processing the remaining vertices of B' . Otherwise, the construction stops. Note that G' is obtained from G_1 by contracting edges of a star forest with centers of the stars contained in A' .

Let us consider the case that the construction stops while processing a vertex $v \in B'$. In that case, $G'[N_v]$ has minimum degree at least $(|N_v| - 1)/2$. Let $N = N_v \cup \{v\}$ and let H be the graph consisting of $G'[N_v]$ and of the vertex v adjacent to all vertices of N_v . Observe that H has minimum degree at least $|N|/2$, and that H is a 1-minor of G . By the choice of B' , we have $|V(H)| = |N| > \frac{c}{4}t^4$. The number of edges of H is at least $|N|^2/4 = \frac{\sqrt{|N|}}{4}|N|^{3/2} \geq \frac{\sqrt{c}}{8}t^2|N|^{3/2} \geq t^2|N|^{3/2}$ edges, since $c \geq 64$. By Lemma 22, H contains K_t' as a subgraph. By Observation 21, this gives a 4-minor of K_t in G . Hence, the second outcome of Lemma 23 holds.

Finally, suppose that all vertices of B' are processed. Recall that each vertex of B' has at least $\frac{c}{4}t^4$ neighbors in A' ; hence, for each vertex of B' , we added at least $\frac{c}{8}t^4$ edges to G' . It follows that G' has at least $\frac{c}{8}t^4|B'| \geq \frac{c}{32}t^4n$ edges. Let us recall that $|V(G')| = |A'| < n^{1-\varepsilon}$, and thus $n \geq |V(G')|^{1/(1-\varepsilon)}$. Consequently, G' is a 1-minor of G with at least

$$\frac{c}{32}t^4n \geq \frac{c}{32}t^4|V(G')|^{1/(1-\varepsilon)} = \frac{c}{32}t^4|V(G')|^{1+\varepsilon+\frac{\varepsilon^2}{1-\varepsilon}} \geq \frac{c}{32}t^4|V(G')|^{1+\varepsilon+\varepsilon^2}$$

edges, as required in the first outcome of Lemma 23. \square

Let us remark that since Lemma 22 gives a 1-subdivision rather than a 1-minor, we can improve the second outcome of Lemma 23 to obtain a 3-minor of K_t rather than a 4-minor, and similarly we could improve other bounds in this section. Nevertheless, the bounds that we obtain in this way still are not likely to be close to optimal, and thus we do not go through this extra effort to improve them.

We now iterate Lemma 23, using Observation 21.

Corollary 24. *Suppose that $m \geq 1$, $t \geq 1$ and $0 < \varepsilon < 1$. If a graph G on n vertices has at least $2 \cdot 32^m t^4 n^{1+\varepsilon}$ edges, then it contains either a graph G' with at least $t^4|V(G')|^{1+\varepsilon+m\varepsilon^2}$ edges as a 4^{m-1} -minor, or K_t as a 4^m -minor.*

By Lemma 22 and Observation 21, the first outcome of Corollary 24 implies the second one when $1 + \varepsilon + m\varepsilon^2 \geq 3/2$. Hence, we obtain the following.

Corollary 25. *Suppose that $0 < \varepsilon \leq 1$ and let $m = \lceil \frac{1}{2\varepsilon^2} \rceil$. If a graph on n vertices has at least $2 \cdot 32^m t^4 n^{1+\varepsilon}$ edges, then it contains K_t as a 4^m -minor.*

Theorem 3 is then easily obtained by carefully choosing the parameters.

Proof of Theorem 3. Let $m = \lceil \frac{18}{\varepsilon^2} \rceil$, $d = 4^m$ and let $n_0 \geq 0$ be the smallest integer such that $n_0^{\varepsilon/6} \geq 2 \cdot 32^m$. Let $t = \lfloor n^{\varepsilon/6} \rfloor$. Then G has at least $n^{1+\varepsilon} \geq 2 \cdot 32^m t^4 n^{1+\varepsilon/6}$ edges and the result follows from Corollary 25. \square

To establish subexponential bounds on expansion, we need another consequence of Corollary 25.

Theorem 26. *Suppose that $2/3 < \delta \leq 1$, $0 < \mu \leq 1$ and $b > 1$. There exists $k_0 \geq 0$ such that for every $k \geq k_0$, if $t = \lfloor e^{\frac{1}{6}k^\delta} \rfloor$ and G is a graph with $n \leq b^k$ vertices and with at least $e^{k^\delta} n$ edges, then G contains K_t as a t^μ -minor.*

Proof. Let $\varepsilon = \frac{1}{6k^{1-\delta} \log b}$, $m = \lceil \frac{1}{2\varepsilon^2} \rceil$ and $k'_0 = (\log 64)^{1/\delta}$. Note that

$$2 \cdot 32^m \leq 64 \cdot 32^{\frac{1}{2\varepsilon^2}} = 64e^{(18 \log 32 \log^2 b)k^{2-2\delta}}. \quad (8)$$

Furthermore, for $k \geq k'_0$, we have

$$\begin{aligned} e^{\frac{1}{6}k^\delta} &\geq 2 \\ t = \lfloor e^{\frac{1}{6}k^\delta} \rfloor &\geq e^{\frac{1}{6}k^\delta} - 1 \geq \frac{1}{2}e^{\frac{1}{6}k^\delta} \geq 1 \\ t^\mu &\geq \frac{1}{2^\mu}e^{\frac{\mu}{6}k^\delta}. \end{aligned} \quad (9)$$

Choose $k_0 \geq k'_0$ large enough that $t^\mu \geq 2 \cdot 32^m$ for every $k \geq k_0$; this is possible by (8) and (9), since $\delta > 2 - 2\delta$.

Since $n \leq b^k$, we have

$$n^\varepsilon \leq b^{\varepsilon k} = b^{\frac{k}{6k^{1-\delta} \log b}} = e^{k^\delta/6},$$

and thus

$$|E(G)| \geq e^{k^\delta} n \geq t^5 e^{k^\delta/6} n \geq t^5 n^{1+\varepsilon} \geq t^\mu t^4 n^{1+\varepsilon} \geq 2 \cdot 32^m t^4 n^{1+\varepsilon}.$$

By Corollary 25, G contains K_t as a 4^m -minor. Note that $4^m < 2 \cdot 32^m \leq t^\mu$ by the choice of k_0 , and thus G contains K_t as a t^μ -minor as required. \square

7 Sublinear separators and expansion

Now, let us turn our attention to sublinear separators. For $\alpha > 0$, a graph G is an α -*expander* if for every $S \subseteq V(G)$ of size at most $|V(G)|/2$, there exist at least $\alpha|S|$ edges of G with exactly one end in S . Random graphs are asymptotically almost surely expanders.

Lemma 27 (Bollobás [4]). *There exists n_0 such that for every even $n \geq n_0$, there exists a 3-regular $\frac{3}{20}$ -expander on n vertices.*

Clearly, expanders do not have sublinear-size separations. This can be extended to their bounded-depth subdivisions.

Lemma 28. *Let $\alpha > 0$ be a real number and let $n, m \geq 1$ be integers. Let G' be obtained from a 3-regular α -expander G on n vertices by subdividing each edge at most m times, and let $n' = |V(G')|$. Any balanced separation in G' has size at least $\frac{n'}{3(1+3m/2)(6/\alpha+2)}$.*

Proof. Let (A', B') be a balanced separation in G' and let $S' = V(A') \cap V(B')$. Note that $n' = |V(A')| + |V(B') \setminus S'| \leq |V(A')| + 2n'/3$, and thus $|V(A')| \geq n'/3$, and similarly $|V(B')| \geq n'/3$. For each $v \in V(G)$, let \tilde{v} denote the corresponding vertex of G' . Let S be a minimal subset of $V(G)$ such that

- for each $v \in V(G)$, if $\tilde{v} \in S'$, then $v \in S$, and
- for every path $P' \subseteq G'$ corresponding to an edge $uv \in E(G)$ such that an internal vertex of P' belongs to S' , we have $\{u, v\} \subseteq S$.

Let $A = \{v \in V(G) \setminus S : \tilde{v} \in V(A')\}$ and $B = \{v \in V(G) \setminus S : \tilde{v} \in V(B')\}$. Note that $|S| \leq 2|S'|$, and that for each two vertices that are connected by a path in $G - S$, the corresponding vertices are also connected by a path in $G' - S'$. Consequently, no vertex of A has a neighbor in B . Without loss of generality, we can assume $|A| \leq n/2$, and since G is an α -expander, it contains at least $\alpha|A|$ edges with one end in A and the other end in S . Since G is 3-regular, we have

$$\alpha|A| \leq 3|S| \leq 6|S'|. \quad (10)$$

Consider a vertex $z \in V(A')$. If $z = \tilde{v}$ for some $v \in V(G)$, then we have $v \in A \cup S$. Similarly, if z is an internal vertex of a path $P' \subseteq G'$ corresponding to an edge $uv \in E(G)$, then $\{u, v\} \subseteq A \cup S$, as otherwise an end of P' would belong to $V(B') \setminus S'$ and P' would contain an internal vertex belonging to S' , contradicting the choice of S . Let H be the subgraph of G induced by $A \cup S$. We observed that each vertex of A' either corresponds to a vertex of H , or it is contained in a path of G' replacing an edge of H . Since H has maximum degree at most 3, it follows that

$$|V(A')| \leq |V(H)| + m|E(H)| \leq |V(H)| + \frac{3}{2}m|V(H)| = (1 + 3m/2)|A \cup S|.$$

Since $|A \cup S| = |A| + |S| \leq (6/\alpha + 2)|S'|$ by (10), we have $(1 + 3m/2)(6/\alpha + 2)|S'| \geq |V(A')| \geq n'/3$. Therefore,

$$|S'| \geq \frac{n'}{3(1 + 3m/2)(6/\alpha + 2)},$$

which gives the lower bound on the size of balanced separations in G' . \square

We are now ready to bound the expansion in small graphs.

Lemma 29. *Let \mathcal{G} be a subgraph-closed class of graphs with strongly sub-linear separators, and let $b > 1$ be a real number. There exists $k_0 \geq 0$ such that for every $k \geq k_0$, every graph $G \in \mathcal{G}$ with at most b^k vertices satisfies $\nabla_k(G) < e^{k^{3/4}}$.*

Proof. Let $c > 0$ and $0 \leq \psi < 1$ be constants such that $s_{\mathcal{G}}(n) \leq cn^\psi$ for every $n \geq 0$. Let $\alpha = \frac{3}{20}$ and let n_0 be a constant such that for every even $n \geq n_0$, there exists a 3-regular α -expander on n vertices (the constant n_0 exists by Lemma 27). Let $k_0 \geq 1$ be large enough so that Theorem 26 applies with $\delta = 3/4$, $\mu = \frac{1-\psi}{2}$ and b ; and furthermore, so that any $k \geq k_0$ and $t = \lfloor e^{\frac{1}{6}k^{3/4}} \rfloor$ satisfies $t \geq n_0 + 1$ and $(t-1)^{1-\psi} > 126c(1+12kt^\mu)$.

Suppose that for some $k \geq k_0$, there exists $G \in \mathcal{G}$ with at most b^k vertices satisfying $\nabla_k(G) \geq e^{k^{3/4}}$. Let G_1 be a k -minor of G with n_1 vertices and at least $e^{k^{3/4}}n_1$ edges. Note that $n_1 \leq |V(G)| \leq b^k$. By Theorem 26, G_1 contains K_t as a t^μ -minor. By Observation 21, G contains K_t as a $(k+t^\mu(2k+1))$ -minor, and thus also as a $4kt^\mu$ -minor. Let G_2 be a 3-regular $\frac{3}{20}$ -expander with either $t-1$ or t vertices, which exists by Lemma 27. Note that G_2 is a $4kt^\mu$ -minor of G , and since G_2 is 3-regular, there exists a graph $G_3 \subseteq G$ obtained from G_2 by subdividing each edge at most $8kt^\mu$ times. Since \mathcal{G} is subgraph-closed, G_3 has a balanced separation of size at most $c|V(G_3)|^\psi$. On the other hand, Lemma 28 implies that every balanced separation in G_3 has size at least $\frac{|V(G_3)|}{3(1+3(8kt^\mu)/2)(6/\alpha+2)} = \frac{|V(G_3)|}{126(1+12kt^\mu)}$. We conclude that

$$c|V(G_3)|^\psi \geq \frac{|V(G_3)|}{126(1+12kt^\mu)},$$

and thus

$$126c(1+12kt^\mu) \geq |V(G_3)|^{1-\psi} \geq (t-1)^{1-\psi}.$$

This is a contradiction by the choice of k_0 . \square

Finally, we combine Lemmas 19, 13 and 29 to obtain the proof of the main result.

Proof of Theorem 2. By Lemma 19, \mathcal{G} is fractionally Vs-fragile, and there exists a constant $b_0 > 1$ such that for any $0 < \varepsilon \leq 1$, the class \mathcal{C}_ε of all graphs in \mathcal{G} such that all their components have at most $b_0^{1/\varepsilon}$ vertices is an ε -witness of the fractional Vs-fragility of \mathcal{G} . Let $b = b_0^8$. Let $k_0 \geq 1$ be the constant of Lemma 29 applied for \mathcal{G} and b . Let $\gamma = 2b^{k_0}$.

For $0 < \varepsilon \leq 1$ and $k \geq 0$, let $g(\varepsilon, k) = \nabla_k(\mathcal{C}_\varepsilon)$. Since \mathcal{C}_ε has bounded component size, it has bounded expansion, and thus $g(\varepsilon, k)$ is finite. Since for every $0 < \varepsilon \leq 1$, every graph in \mathcal{G} has a fractional \mathcal{C}_ε -complementary packing of thickness at most ε , the choice of g implies that \mathcal{G} is fractionally (Be, g) -fragile.

Let G' be any graph of $\mathcal{C}_{1/(4k+4)}$. Note that each component of G' has at most $b_0^{4k+4} \leq b^k$ vertices, and thus if $k \leq k_0$, we have $\nabla_k(G') \leq b^{k_0}$. On the other hand, if $k > k_0$, then Lemma 29 (applied to each component separately) implies that $\nabla_k(G') < e^{k^{3/4}}$. In both cases, we have $\nabla_k(G') \leq b^{k_0} e^{k^{3/4}}$. It follows that

$$g\left(\frac{1}{4k+4}, k\right) = \nabla_k(\mathcal{C}_{1/(4k+4)}) \leq b^{k_0} e^{k^{3/4}}.$$

By Lemma 13, we have

$$\nabla_k(\mathcal{G}) \leq 2g\left(\frac{1}{4k+4}, k\right) \leq 2b^{k_0} e^{k^{3/4}} = \gamma e^{k^{3/4}}$$

for every integer $k \geq 0$, as required. \square

References

- [1] B. BAKER, *Approximation algorithms for NP-complete problems on planar graphs*, Journal of the ACM (JACM), 41 (1994), pp. 153–180.
- [2] C. BAZGAN, B. ESCOFFIER, AND V. T. PASCHOS, *Completeness in standard and differential approximation classes: Poly-(D)APX-and (D)PTAS-completeness*, Theoretical Computer Science, 339 (2005), pp. 272–292.
- [3] E. BERGER, Z. DVOŘÁK, AND S. NORIN, *Treewidth of 3-dimensional grid partitions*. In preparation, 2014.
- [4] B. BOLLOBÁS, *The isoperimetric number of random regular graphs*, European Journal of Combinatorics, 9 (1988), pp. 241–244.
- [5] B. COURCELLE, *The monadic second-order logic of graphs. I. Recognizable sets of finite graphs*, Information and computation, 85 (1990), pp. 12–75.
- [6] V. G. DEI, B. KLINZ, G. J. WOEGINGER, ET AL., *Exact algorithms for the Hamiltonian cycle problem in planar graphs*, Operations Research Letters, 34 (2006), pp. 269–274.

- [7] E. DEMAINE, M. HAJIAGHAYI, AND K. KAWARABAYASHI, *Contraction decomposition in H -minor-free graphs and algorithmic applications*, in Proceedings of the Forty-third Annual ACM Symposium on Theory of Computing, STOC '11, ACM, 2011, pp. 441–450.
- [8] M. DeVOS, G. DING, B. OPOROWSKI, D. SANDERS, B. REED, P. SEYMOUR, AND D. VERTIGAN, *Excluding any graph as a minor allows a low tree-width 2-coloring*, J. Comb. Theory, Ser. B, 91 (2004), pp. 25–41.
- [9] R. DOWNEY AND M. FELLOWS, *Fixed-parameter tractability and completeness. II. On completeness for $W[1]$* , Theoretical Computer Science, 141 (1995), pp. 109–131.
- [10] P. G. DRANGE, M. S. DREGI, F. V. FOMIN, S. KREUTZER, D. LOKSHTANOV, M. PILIPCZUK, M. PILIPCZUK, F. REIDL, S. SAURABH, F. S. VILLAAMIL, AND S. SIKDAR, *Kernelization and sparseness: the case of dominating set*, arXiv e-prints, 1411.4575 (2014).
- [11] Z. DVOŘÁK, *Asymptotical structure of combinatorial objects*, PhD thesis, Charles University in Prague, 2007.
- [12] —, *On forbidden subdivision characterizations of graph classes*, European Journal of Combinatorics, 29 (2008), pp. 1321–1332.
- [13] Z. DVOŘÁK, D. KRÁL, AND R. THOMAS, *Testing first-order properties for subclasses of sparse graphs*, Journal of the ACM (JACM), 60 (2013), p. 36.
- [14] Z. DVOŘÁK AND S. NORIN, *Treewidth of graphs with balanced separations*, ArXiv e-prints, 1408.3869 (2014).
- [15] Z. DVOŘÁK AND S. NORINE, *Small graph classes and bounded expansion*, Journal of Combinatorial Theory, Series B, 100 (2010), pp. 171–175.
- [16] D. EPPSTEIN, *Diameter and treewidth in minor-closed graph families*, Algorithmica, 27 (2000), pp. 275–291.
- [17] D. EPPSTEIN, Z. GALIL, G. F. ITALIANO, AND T. H. SPENCER, *Separator based sparsification: I. Planarity testing and minimum spanning trees*, Journal of Computer and System Sciences, 52 (1996), pp. 3–27.

- [18] ———, *Separator-based sparsification II: Edge and vertex connectivity*, SIAM Journal on Computing, 28 (1998), pp. 341–381.
- [19] G. N. FEDERICKSON, *Fast algorithms for shortest paths in planar graphs, with applications*, SIAM Journal on Computing, 16 (1987), pp. 1004–1022.
- [20] M. GAREY AND D. JOHNSON, *Computers and Intractability: A Guide to the Theory of NP-completeness*, WH Freeman & Co. New York, NY, USA, 1979.
- [21] J. GILBERT, J. HUTCHINSON, AND R. TARJAN, *A separator theorem for graphs of bounded genus*, Journal of Algorithms, 5 (1984), pp. 391–407.
- [22] M. T. GOODRICH, *Planar separators and parallel polygon triangulation*, Journal of Computer and System Sciences, 51 (1995), pp. 374–389.
- [23] T. JIANG, *Compact topological minors in graphs*, J. Graph Theory, 67 (2011), pp. 139–152.
- [24] K. KAWARABAYASHI AND B. REED, *A separator theorem in minor-closed classes*, in Proc. 51st Annual IEEE Symposium on. Foundations of Computer Science, 2010.
- [25] P. N. KLEIN, S. MOZES, AND O. WEIMANN, *Shortest paths in directed planar graphs with negative lengths: A linear-space $O(n \log_2 n)$ -time algorithm*, ACM Transactions on Algorithms (TALG), 6 (2010), p. 30.
- [26] J. KOMLÓS AND E. SZEMERÉDI, *Topological cliques in graphs II*, Comb. Probab. Comput., 5 (1996), pp. 79–90.
- [27] A. KOSTOCHKA AND L. PYBER, *Small topological complete subgraphs of "dense" graphs*, Combinatorica, 8 (1988), pp. 83–86.
- [28] R. LIPTON AND R. TARJAN, *A separator theorem for planar graphs*, SIAM Journal on Applied Mathematics, 36 (1979), pp. 177–189.
- [29] ———, *Applications of a planar separator theorem*, SIAM Journal on Computing, 9 (1980), pp. 615–627.
- [30] J. MATOUŠEK AND J. VONDRÁK, *The probabilistic method*. <http://kam.mff.cuni.cz/~matousek/lectnotes.html>. Lecture notes.

- [31] J. NEŠETŘIL AND P. OSSONA DE MENDEZ, *Grad and classes with bounded expansion I. Decomposition.*, European J. Combin., 29 (2008), pp. 760–776.
- [32] ———, *Grad and classes with bounded expansion II. Algorithmic aspects.*, European J. Combin., 29 (2008), pp. 777–791.
- [33] ———, *Sparsity – Graphs, Structures, and Algorithms*, Springer, 2012.
- [34] J. NEŠETŘIL AND P. OSSONA DE MENDEZ, *Induced matchings and induced paths in graphs*, ITI series, 335 (2007).
- [35] J. NEŠETŘIL, P. OSSONA DE MENDEZ, AND D. WOOD, *Characterisations and examples of graph classes with bounded expansion*, Eur. J. Comb., 33 (2012), pp. 350–373.
- [36] J. NEŠETŘIL AND S. POLJAK, *Complexity of the subgraph problem*, Comment. Math. Univ. Carol., 26 (1985), pp. 415–420.
- [37] S. PLOTKIN, S. RAO, AND W. D. SMITH, *Shallow excluded minors and improved graph decompositions*, in Proceedings of the fifth annual ACM-SIAM symposium on Discrete algorithms, Society for Industrial and Applied Mathematics, 1994, pp. 462–470.
- [38] N. ROBERTSON AND P. D. SEYMOUR, *Graph Minors. III. Planar tree-width*, J. Combin. Theory, Ser. B, 36 (1984), pp. 49–64.
- [39] N. ROBERTSON AND P. D. SEYMOUR, *Graph minors. II. Algorithmic aspects of tree-width*, Journal of algorithms, 7 (1986), pp. 309–322.
- [40] A. THOMASON, *An extremal function for complete subgraphs*, Math. Proc. Camb. Phil. Soc., 95 (1984), pp. 261–265.
- [41] C. WULFF-NILSEN, *Girth of a planar digraph with real edge weights in $O(n(\log n)^3)$ time*, arXiv e-prints, 0908.0697 (2009).